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Oscillations of a rotating liquid drop

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The effect of rotation on the frequencies of oscillations of a liquid drop is investigated. It is assumed that the drop is imbedded in a fluid of the same or different density and that a constant surface tension acts on the interface. Rotation influences the oscillations through the Coriolis force and through the centrifugal distortion of the drop. For non-axisymmetric oscillations only the Coriolis force is important in first approximation and causes the expected splitting of the frequency for the two modes differing in their sign of circular polarization with respect to the axis of rotation. In the case of axisymmetric oscillations the centrifugal distortion and the Coriolis force combine to increase the frequency whenever the density ρ^{i} of the drop exceeds the density of ρ^{o} of the surrounding fluid. For $\rho^{\mathrm{i}} < \rho^{\mathrm{o}}$ a decrease of the frequency of oscillation is possible for some modes of higher degree.

1. Introduction

The oscillations of a rotating liquid drop represent one of the basic phenomena in fluid mechanics which have wide-ranging applications. Rotating as well as non-rotating liquid drops with surface tension have been used as models of the atomic nucleus, and close analogies can be drawn between the excited states of nuclei and the oscillations of liquid drops (Swiatecki 1974). Since surface tension has a similar dynamic effect as gravity, rotating drops have been used as simple models of stars, and Chandrasekhar (1965) has shown that the equilibrium states of rotating drops and their stability properties resemble those of rotating self-gravitating masses.

The interest in rotating drops has been revived in recent years by the striking drop experiments performed in the gravity-free environment of Skylab. Because of their large size these drops exhibited surface-tension-induced oscillations with frequencies only slightly in excess of their rotation rates. The Coriolis force must thus be taken into account in understanding the dynamics of these drops. Experiments in Earthbound laboratories are more difficult to perform since the drop must be suspended in a liquid of nearly the same density. For this reason no quantitative data on the oscillations of rotating liquid drops are yet available in the literature. Only recently have experimental measurements been started (Wang & Trinh 1984), which have in part motivated the present study.

The lack of detailed experimental data on the oscillation frequencies of rotating drops may have caused a delay of a general theoretical analysis of the problem. Rosenkilde (1967) treats the problem using the method of tensor virials, but he does not derive explicit expressions for the frequencies of oscillations. Because of the assumption of distortions of the rotating drop in the form of pure Legendre functions, an accurate determination of the oscillation frequency cannot be achieved by this approach. Recently the finite-amplitude oscillations of a rotating cylindrical drop have been analysed numerically (Benner, Patzen & Scriven 1982). In this paper the more restricted goal of the linear theory of oscillations of a nearly spherical rotating drop will be pursued.

Several analytical approaches to the problem of oscillations of a rotating drop appear to be possible. The problem can be considered as an extension of the problem of inertial oscillations of a rotating spheroidal cavity (Kudlick 1966). By introducing surface tension and by allowing for surface distortions, the changes in the frequency of inertial modes due to surface-tension effects as well as the effects of rotation on the surface-tension modes can be calculated. The difference between the shape of a rotating drop and the ellipsoidal figure for which analytical solutions for inertial oscillations exist will cause inaccuracies, however, unless the centrifugal force is small compared with the surface tension divided by the surface area of the drop. Moreover, special difficulties are encountered when the dynamics of the outside medium is taken into account. Since Earthbound experiments require suspension of the drop in a fluid with closely matched density, the problem of inertial oscillations in the outside medium must be considered. No analytical solutions for this problem exist.

In view of these difficulties, an alternative analytical method is employed in this paper. Instead of considering the modifications of inertial oscillations due to surface-tension effects, only the influence of the rotation rate Ω on the surface-tension-induced oscillations is calculated. By introducing rotation as perturbation into the problem, the difficulty caused by inertial oscillations of the outside medium can be avoided. Comparison with the recently performed experimental observations (Wang & Trinh 1984) indicates that this method of analysis yields results even when the rotation rate becomes comparable to the frequency of drop oscillations. But the method is not capable of determining the influence of surface tension on inertial modes whose frequencies ω are restricted to the range $-2\Omega < \omega < 2\Omega$ (Greenspan 1968).

The effect of rotation on non-axisymmetric modes of oscillations is of first order in the rotation rate Ω and can easily be calculated since the influence of the centrifugal force is negligible. The results are similar to those obtained for the rotational splitting of the free oscillations of the Earth obtained by Backus & Gilbert (1961) and Pekeris, Alterman & Zarosch (1961). The focus of the present analysis is on the axisymmetric modes for which the effects of rotation are of the order Ω^2 and the centrifugal force can no longer be neglected. It is still possible, however, to obtain analytical expressions for the change of the frequency of oscillation. Viscous damping is neglected since its effects on the frequency ω of oscillation is small as long as $\nu/r_0^2 \ll \omega$, where ν is the kinematic viscosity of the drop and r_0 is its mean radius (Prosperetti 1980). It seems possible to calculate the effect of rotation on the decay rate of oscillations following the analytical theory developed by Prosperetti in the case of the non-rotating drop, but this task will not be attempted in the present paper.

2. Equilibrium figure of a rotating drop

The problem of the equilibrium shape of a liquid drop in the state of rigid rotation has been treated by a number of authors going back to Poincaré (1895). A succinct treatment of the problem is given in Chandrasekhar's (1965) paper. Here the problem is described only briefly in order to provide a basis for the analysis of oscillations.

The pressure p^{i} inside a homogeneous rotating drop of density ρ^{i} is given by

$$p^{\mathrm{i}} = p_0^{\mathrm{i}} + \tfrac{1}{2}\rho^{\mathrm{i}} r^2 (1-\cos^2\theta)\,\Omega^2, \tag{2.1a}$$

while the pressure in the homogeneous outside medium rotating with the same angular velocity Ω as the drop obeys the analogous relationship

$$p^{\mathrm{o}} = p_0^{\mathrm{o}} + \tfrac{1}{2}\rho^{\mathrm{o}} r^2 (1-\cos^2\theta)\,\Omega^2. \tag{2.1b}$$

A spherical coordinate system (r, θ, ϕ) has been assumed, with the origin at the centre of the drop and with the polar axis in the direction of the axis of rotation. The axisymmetric equilibrium shape of the rotating drop is given by

$$0 = F_0(r,\theta) \equiv r^3 - r_0^3[1-\epsilon P_2(\cos\theta) + \epsilon^2 \sum_{n>2} a_n P_n(\cos\theta)], \tag{2.2}$$

where r_0 refers to the mean radius of the drop, i.e. the volume of the drop is given by $\frac{4}{3}\pi r_0^3$. Since the shape of the drop is purely spheroidal for small rates of rotation, the amplitude ϵ of the P_2 component of the Legendre-function expansion in (2.2) has been introduced as a parameter of the problem.

The shape is determined by the static equilibrium condition at the interface of the two fluids:

$$p^{\mathrm{i}} - p^{\mathrm{o}} = T\nabla\cdot\boldsymbol{n}_0 \quad \text{at} \quad F_0 = 0, \tag{2.3}$$

where T is the surface or interfacial tension and $\mathbf{n}_0$ is the normal unit vector of the surface:

$$\boldsymbol{n}_0 = \nabla F_0/|\nabla F_0|. \tag{2.4}$$

Evaluation of $\nabla\cdot\boldsymbol{n}_0$ at the interface $F_0 = 0$ yields

$$\nabla\cdot\boldsymbol{n}_0|_{F_0=0} = r_0^{-1}\left\{2 - \frac{4\epsilon}{3}P_2(\cos\theta) - \tfrac{20}{9}\epsilon^2(P_2(\cos\theta))^2 + \tfrac{2}{9}\epsilon^2\left(\frac{\partial}{\partial\theta}P_2\right)^2 \right.$$
$$\left. + \epsilon^2 \sum_{n>2} \frac{n(n+1)-2}{3} a_n P_n(\cos\theta) + O(\epsilon^3)\right\}, \tag{2.5}$$

where terms of the order ϵ^3 have not been written explicitly. By insertion of (2.5) in (2.3), the equilibrium shape can easily be determined up to terms of order ϵ^3:

$$\epsilon = S(\rho^{\mathrm{i}} - \rho^{\mathrm{o}}) + \tfrac{1}{7}S^2(\rho^{\mathrm{i}} - \rho^{\mathrm{o}})^2 + \ldots, \quad a_4 = \tfrac{12}{35} \text{ all other } a_n = 0, \tag{2.6}$$

where

$$S \equiv \Omega^2 r_0^3/4T. \tag{2.7}$$

For the calculations performed in the following only terms of order ϵ will actually be needed.

3. The mathematical problem of small-amplitude oscillations

The equation for the small deviation $\rho\Phi$ of the pressure from its hydrostatic value in a rotating inviscid incompressible fluid is given by

$$\nabla^2\phi = \left(\frac{2\Omega}{\omega}\right)^2 (\boldsymbol{k}\cdot\nabla)^2\,\Phi, \tag{3.1}$$

where a time dependence of the form $\exp\{\mathrm{i}\omega t\}$ has been assumed. The unit vector $\boldsymbol{k}$ points in the direction of the axis of rotation. The velocity field $\boldsymbol{v}$ corresponding to Φ is given by

$$\boldsymbol{v} = \frac{+\mathrm{i}\omega\nabla\Phi - 2\Omega\boldsymbol{k}\times\nabla\Phi - \mathrm{i}\boldsymbol{k}(2\Omega)^2\,\boldsymbol{k}\cdot\nabla\Phi/\omega}{\omega^2 - 4\Omega^2}. \tag{3.2}$$

For a derivation of (3.1) and (3.2) see Greenspan (1968). For the problem of linear oscillations of the rotating drop, (3.1) must be solved for the dynamic pressures $\rho^{\mathrm{i}}\Phi^{\mathrm{i}}$,

$\rho^o\Phi^o$ inside and outside the interface. For small distortions ζ in the normal direction the position of the interface can be described by

$$F = F_0 - \zeta|\nabla F_0| = 0. \tag{3.3}$$

The condition for the change in pressure across the interface is given by

$$p^i + \rho^i\Phi^i - (p^o + \rho^o\Phi^o) = T\nabla\cdot\boldsymbol{n} \quad \text{at} \quad F = 0, \tag{3.4}$$

where p^i, p^o refer to the hydrostatic pressure distributions (2.1), and $\boldsymbol{n}$ is given by $\nabla F/|\nabla F|$. The description of the problem is completed by the kinematic relationship

$$-\mathrm{i}\omega\zeta|\nabla F_0| + \boldsymbol{v}\cdot\nabla F_0 = 0. \tag{3.5}$$

The problem of oscillations can readily be solved in the non-rotating case $\Omega = 0$. Assuming a distortion of the interface given by

$$\zeta_0 = 3r_0^3\alpha_n P_n^m(\cos\theta)\exp\{\mathrm{i}m\phi\}|\nabla F_0|^{-1}, \tag{3.6}$$

the condition (3.4) can be written in the form

$$\rho^i\Phi_0^i - \rho^o\Phi_0^o = r_0^{-1}T\alpha_n P_n^m(\cos\theta)\exp\{\mathrm{i}m\phi\}[n(n+1)-2] \quad \text{at} \quad r = r_0. \tag{3.7}$$

For simplicity we are neglecting the common factor $\exp\{\mathrm{i}\omega t\}$ in the definition of ζ_0, $\phi_0^{i,o}$ and $\boldsymbol{v}_0$. The r-dependence of ζ_0 has been chosen such that the distortion does not cause a change in the volume of the drop. Since the Laplacian of $\Phi_0^{i,o}$ must vanish, the solutions exhibiting the asymmetry of the distortion (3.6) can be written in the form

$$\Phi_0^i = \left(\frac{r}{r_0}\right)^n \beta_n^i P_n^m(\cos\theta)\exp\{\mathrm{i}m\phi\}, \tag{3.8a}$$

$$\Phi_0^o = \left(\frac{r_0}{r}\right)^{n+1} \beta_n^o P_n^m(\cos\theta)\exp\{\mathrm{i}m\phi\}. \tag{3.8b}$$

Using

$$\boldsymbol{v}_0 = \mathrm{i}\nabla\Phi_0/\omega_0, \tag{3.9}$$

we obtain from relationship (3.5)

$$-\omega_0^2\alpha_n = -\frac{n\beta_n^i}{r_0^2} = \frac{(n+1)\beta_n^o}{r_0^2}. \tag{3.10}$$

Condition (3.7) can now be satisfied by the choice

$$\omega_0^2 = Tr_0^{-3}(n-1)(n+2)\left[\frac{\rho^i}{n} + \frac{\rho^o}{n+1}\right]^{-1}, \tag{3.11}$$

which is the well-known expression for the frequencies of oscillations of a non-rotating drop (Lamb 1932, p. 475).

For non-axisymmetric oscillations the above calculation can easily be extended to take into account the Coriolis force to first order in Ω. Since the drop shape remains spherical to this order and the dynamic pressure Φ is still a harmonic function, the only change occurs in the relationship between $\boldsymbol{v}$ and Φ. Instead of (3.9) we obtain

$$\tilde{\boldsymbol{v}} = \frac{\mathrm{i}\nabla\Phi_0}{\tilde{\omega}} - \frac{2\Omega\boldsymbol{k}\times\nabla\Phi_0}{\tilde{\omega}^2}. \tag{3.12}$$

Accordingly the relationships (3.10) become dependent on the azimuthal wave-number m:

$$-\tilde{\omega}^2\alpha_n = -\beta_n^i\left(n + \frac{2\Omega}{\tilde{\omega}}m\right)\Big/r_0^2 = \beta_n^o\left(n+1-2\frac{\Omega}{\tilde{\omega}}m\right)\Big/r_0^2. \tag{3.13}$$

To first approximation in Ω the frequencies $\tilde{\omega}$ of non-axisymmetric oscillations of a rotating drop are thus given by

$$\tilde{\omega}^2 = \omega_0^2\left[1+\frac{2\Omega m\omega_0^{-1}(\rho^{\mathrm{i}}(n+1)^2-\rho^{\mathrm{o}}n^2)}{n(n+1)\,(\rho^{\mathrm{i}}(n+1)+n\rho^{\mathrm{o}})}\right]. \tag{3.14}$$

Axisymmetric oscillations depend only on even powers of Ω since the term proportiontal to Ω in (3.2) does not enter the analysis. This property suggests an expansion of the solution for axisymmetric oscillations in powers of the parameter $\eta \equiv 4\Omega^2/\omega_0^2$:

$$\left.\begin{aligned} \boldsymbol{\Phi} &= [\Phi_0+\eta\Phi_1+\ldots]\exp\{\mathrm{i}\omega t\}, \quad \boldsymbol{v} = [\boldsymbol{v}_0+\eta\boldsymbol{v}_1\ldots]\exp\{\mathrm{i}\omega t\},\\ \omega &= \omega_0+\eta\omega_1+\ldots, \quad \zeta = [\zeta_0+\eta\zeta_1+\ldots]\exp\{\mathrm{i}\omega t\}. \end{aligned}\right\} \tag{3.15}$$

The functions with subscript 0 and ω_0 are given by the above expressions (3.8), (3.9) and (3.11) with $m = 0$. In §4 the problem will be considered to order η.

4. The effect of rotation on axisymmetric oscillations

Equations (3.1) and (3.2) yield the following equations to order η:

$$\nabla^2\Phi_1 = (\boldsymbol{k}\cdot\nabla)^2\,\Phi_0, \tag{4.1}$$

$$\boldsymbol{v}_1^* = \frac{\mathrm{i}}{\omega_0}\left(\nabla\Phi_1+(1-\boldsymbol{kk}\cdot)\,\nabla\Phi_0-\frac{\omega_1}{\omega_0}\nabla\Phi_0\right). \tag{4.2}$$

Both equations must be satisfied in the inner as well as in the outer region of the problem. The asterisk serves as a reminder that only the meridional component of the velocity field is given by (4.2). The relationship (3.5) for the distortion yields

$$\begin{aligned} -\mathrm{i}\omega_0\,\zeta_1+\boldsymbol{v}_1\cdot\boldsymbol{n}_0 &= \frac{\omega_1}{\omega_0}\boldsymbol{v}_0\cdot\boldsymbol{n}_0-\xi\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\theta}\Phi_0\frac{\partial}{\partial\theta}\frac{P_2(\cos\theta)}{3r_0}\\ &\quad+\mathrm{i}\xi P_2(\cos\theta)\left[r_0\frac{\partial^2}{\partial r^2}\Phi_0+2\frac{\partial}{\partial r}\Phi_0\right]\Big/3\omega_0 \quad \text{at} \quad r = r_0, \end{aligned} \tag{4.3}$$

where the parameter ξ denotes the ratio ϵ/η, which to order η is given by

$$\xi \equiv \frac{\epsilon}{\eta} = \frac{\rho^{\mathrm{i}}-\rho^{\mathrm{o}}}{(n+1)\,\rho^{\mathrm{i}}+n\rho^{\mathrm{o}}}\,\frac{(n-1)\,n(n+1)\,(n+2)}{16}. \tag{4.4}$$

The last term on the right-hand side of (4.3) arises because the solution of zeroth order in η satisfies (3.5) at $r = r_0$ instead at $F_0 = 0$. Similarly, additional inhomogeneous terms are generated when the condition (3.4) for the normal stress is evaluated:

$$\begin{aligned} \rho^{\mathrm{i}}\Phi_1^{\mathrm{i}}-\rho^{\mathrm{o}}\Phi_1^{\mathrm{o}}-\tfrac{1}{3}\xi r_0\,P_2(\cos\theta)\frac{\partial}{\partial r}[\rho^{\mathrm{i}}\Phi_0^{\mathrm{i}}-\rho^{\mathrm{o}}\Phi_0^{\mathrm{o}}]+\tfrac{1}{4}\alpha_n P_n(\cos\theta)\,r_0^2(1-\cos^2\theta)\,(\rho^{\mathrm{i}}-\rho^{\mathrm{o}})\,\omega_0^2\\ = \frac{T}{r_0}\{\sum_\nu \gamma_\nu(\nu+2)\,(\nu-1)\,P_\nu(\cos\theta)+\alpha_n\,\xi F_n(\theta)\}. \end{aligned} \tag{4.5}$$

In writing this relationship the definitions

$$\zeta_1 \equiv r_0\sum_\nu \gamma_\nu P_\nu(\cos\theta), \tag{4.6a}$$

$$F_n(\theta) \equiv \frac{4}{3}(n^2+n+4)\,P_2(\cos\theta)\,P_n(\cos\theta)-\frac{4}{3}\frac{\partial}{\partial\theta}P_2\frac{\partial}{\partial\theta}P_n \tag{4.6b}$$

have been introduced.

There is no need to solve the problem defined by (4.1)–(4.3) and (4.5) explicitly, since ω_1 is the quantity of primary interest. This quantity is determined by the solvability condition for the problem. By multiplying (4.1) in the outside region by $\rho^{o}\Phi_0^{o}$ and in the inside region by $\rho^{o}\Phi_0^{i}$, integrating the equations over the respective regions and adding them, we obtain after partial integrations

$$\rho^{o}\int_{o}\phi_0^{o}(\boldsymbol{k}\cdot\nabla)^2\Phi_0^{o}\,\mathrm{d}^3\tau+\rho^{i}\int_{i}\Phi_0^{i}(\boldsymbol{k}\cdot\nabla)^2\phi_0^{i}\,\mathrm{d}^3\tau$$
$$=\oint\left\{\rho^{i}\left[\Phi_0^{i}\frac{\partial}{\partial r}\Phi_1^{i}-\Phi_1^{i}\frac{\partial}{\partial r}\Phi_0^{i}\right]-\rho^{o}\left[\Phi_0^{o}\frac{\partial}{\partial r}\Phi_1^{o}-\Phi_1^{o}\frac{\partial}{\partial r}\Phi_0^{o}\right]\right\}\mathrm{d}^2\sigma, \quad (4.7)$$

where the surface integral is extended over the surface $r=r_0$. Because of the relationships

$$\boldsymbol{k}\cdot\nabla\, r^nP_n=nr^{n-1}P_{n-1},\quad \boldsymbol{k}\cdot\nabla r^{-n-1}P_n=-(n+1)\,r^{-n-2}P_{n+1}, \quad (4.8)$$

the left-hand side of (4.7) vanishes. The right-hand side yields an expression for ω_1 after the dependence on the functions Φ_1^{i}, Φ_1^{o} is eliminated. For this purpose we first obtain a relationship for ζ_1 from (4.2), (4.3):

$$\omega_0^2\zeta_1=\frac{\partial}{\partial r}\Phi_1^{i,o}+\left(1-2\frac{\omega_1}{\omega_0}\right)\frac{\partial}{\partial r}\Phi_0^{i,o}-\tfrac{1}{3}\xi\left\{P_2(\cos\theta)\frac{\partial^2}{\partial r^2}(r\Phi_0^{i,o})\right.$$
$$\left.-\frac{\partial}{\partial\theta}P_2\frac{\partial}{\partial\theta}\frac{\Phi_0^{i,o}}{r_0}\right\}-\cos\theta\,\boldsymbol{k}\cdot\nabla\Phi_0^{i,o}\quad\text{at}\quad r=r_0. \quad (4.9)$$

Only the $P_n(\cos\theta)$-component of ζ_1 in (4.6) is needed, since $\partial\Phi_1^{i,o}/\partial r$ is multiplied by $\Phi_0^{i,o}$ in the surface integral of (4.7). After using (4.3) and the analogous relationship of zeroth order for expressing $\partial\Phi_0^{i,o}/\partial r$ in terms of $\Phi_0^{i,o}$, we thus obtain from (4.7)

$$0=\frac{1}{r_0(\rho^{i}/n+\rho^{o}/(n+1))}\oint\{(\rho^{i}\Phi_0^{i}-\rho^{o}\Phi_0^{o})\,(\rho^{i}\Phi_1^{i}-\rho^{o}\Phi_1^{o})$$
$$-(\rho^{i}\Phi_1^{i}-\rho^{o}\Phi_1^{o})\,(\rho^{i}\Phi_0^{i}-\rho^{o}\Phi_0^{o})\}\,\mathrm{d}^2\sigma+\left(2\frac{\omega_1}{\omega_0}-1\right)\oint\left\{\rho^{i}\Phi_0^{i}\frac{\partial}{\partial r}\Phi_0^{i}-\rho^{o}\Phi_0^{o}\frac{\partial}{\partial r}\Phi_0^{o}\right\}\mathrm{d}^2\sigma$$
$$-\oint\left\{(\rho^{i}\Phi_0^{i}-\rho^{o}\Phi_0^{o})\left[\tfrac{1}{3}\xi P_2(\cos\theta)\frac{\partial}{\partial r}(\rho^{i}\Phi_0^{i}-\rho^{o}\Phi_0^{o})+r_0^{-2}\alpha_n\right.\right.$$
$$\left.\left.\xi T F_n(\theta)-\tfrac{1}{4}\alpha_n\,P_n(\cos\theta)\,r_0\,\omega_0^2(\rho^{i}-\rho^{o})\,(1-\cos^2\theta)\right]\right\}\mathrm{d}^2\sigma\left(\frac{\rho^{i}}{n}+\frac{\rho^{o}}{n+1}\right)^{-1}$$
$$+\oint\left\{\rho^{i}\Phi_0^{i}\left(\tfrac{1}{3}\xi P_2\frac{\partial^2}{\partial r^2}(r\Phi_0^{i})-\frac{\partial}{\partial\theta}P_2\frac{\partial}{\partial\theta}\frac{\Phi_0^{i}\xi}{3r_0}+\cos\theta\,\boldsymbol{k}\cdot\nabla\phi_0^{i}\right)-\rho^{o}\phi_0^{o}\right.$$
$$\left.\left(\tfrac{1}{3}\xi P_2\frac{\partial^2}{\partial r^2}(r\Phi_0^{i})-\frac{\partial}{\partial\theta}P_2\frac{\partial}{\partial\theta}\frac{\Phi_0^{o}\xi}{3r_0}+\cos\theta\,\boldsymbol{k}\cdot\nabla\Phi_0^{o}\right)\right\}\mathrm{d}^2\sigma. \quad (4.10)$$

The first integral on the right-hand side vanishes identically, and the remaining terms yield the following expression for ω_1:

$$\eta\frac{\omega_1}{\omega_0}=\frac{\Omega^2r_0^3}{T}\left\{\frac{2\rho^{i}}{n(n+2)\,(2n-1)}+\frac{2\rho^{o}}{(n^2-1)\,(2n+3)}+\frac{\rho^{i}-\rho^{o}}{6(2n-1)\,(2n+3)}\right.$$
$$\left.\times\left[\frac{n^4+2n^3-4n^2-5n+6}{(n+2)\,(n-1)}-\frac{\rho^{o}(n+4)\,n^2+\rho^{i}(n-3)\,(n+1)^2}{4(\rho^{i}(n+1)+\rho^{o}n)}\right]\right\}. \quad (4.11)$$

In obtaining this expression it is convenient to use the formula

$$\int_0^\pi P_n(\cos\theta)\frac{\partial}{\partial\theta}P_n(\cos\theta)\frac{\partial}{\partial\theta}P_2(\cos\theta)\sin\theta\,d\theta = 3\int_0^\pi P_2(\cos\theta)\,[P_n(\cos\theta)]^2\sin\theta\,d\theta$$
$$= \frac{6(n+1)\,n}{(2n+3)\,(2n+1)\,(2n-1)}. \quad (4.12)$$

According to (3.15), (4.11) describes the relative change of the frequencies of axisymmetric modes due to rotation in first approximation.

5. Discussion

In (4.11) the influence of the rotational distortion of the drop and the effect of the Coriolis force can be distinguished by their different dependence on the density difference between the drop and the outside fluid. When this difference vanishes, $\rho^{\mathrm{i}} = \rho^{\mathrm{o}}$, the drop remains spherical and the increase of the frequency of oscillation must be attributed solely to the Coriolis force. As the rate of rotation increases from zero, the velocity field changes from a potential flow to a flow with finite vorticity. The change in the azimuthal velocity induced by the requirement that angular momentum be conserved during the oscillation provides the additional restoring force that is responsible for the increase of the frequency of oscillation.

When the density of the drop differs from that of the outside fluid, the shape of the drop is distorted owing to the centrifugal force into an oblate or prolate spheroid. In the oblate case, $\rho^{\mathrm{i}} > \rho^{\mathrm{o}}$, a further increase of the frequency of oscillations results from the enhanced effect of the centrifugal restoring force. The opposite effect occurs for $\rho^{\mathrm{i}} < \rho^{\mathrm{o}}$, and for higher values of n even a decrease of the frequency of oscillations with increasing rate of rotation is possible if $\rho^{\mathrm{o}}/\rho^{\mathrm{i}}$ is sufficiently large.

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REFERENCES

Benner, R. E., Patzek, T. W. & Scriven, L. E. 1982 Nonlinear oscillations of translationally-symmetric, inviscid rotating drops. *Bull. Am. Phys. Soc.* **27**, 1168.

Backus, G. E. & Gilbert, F. 1961 The rotational splitting of the free oscillations of the earth. *Proc. Natl Acad. Sci.* **47**, 362–371.

Chandrasekhar, S. 1965 The stability of a rotating liquid drop. *Proc. R. Soc. Lond.* A **286**, 1–26.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

Kudlick, M. D. 1966 On transient motions in a contained rotating fluid. Ph.D. thesis, Mathematics Dept, MIT.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Pekeris, G. L., Alterman, Z. & Jarosch, H. 1961 Comparison of theoretical with observed values of the periods of free oscillations of the earth. *Proc. Natl Acad. Sci.* **47**, 91–98.

Poincaré, H. 1895 *Capilarité*. Paris: George Garre.

Prosperetti, A. 1980 Normal-mode analysis for the oscillations of a viscous liquid drop in an immiscible liquid. *J. Méc.* **19**, 149–182.

ROSENKILDE, C. E. 1967 Stability of axisymmetric figures of equilibrium of a rotating charged liquid drop. *J. Math. Phys.* **8**, 98–118.

SWIATECKI, W. J. 1974 The rotating, charged or gravitating liquid drop, and problems in nuclear physics and astronomy. In *Proc. Intl Colloq. on Drops and Bubbles* (ed. D. J. Collins, M. S. Plesset & M. M. Saffren), pp. 52–78. California Institute of Technology and Jet Propulsion Laboratory.

WANG, T. G. & TRINH, E. 1984 Study of drop oscillation and rotation in immiscible liquid systems. In *Proc. Symp. on Variational Methods for Equilibrium Problems in Fluids, June 1983, Trento, Italy.*